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On spectral integral variations of mixed graphs ☆

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Abstract

In this paper, we characterize the mixed graphs with exactly one Laplacian eigenvalue moving up by an integer and other Laplacian eigenvalues remaining invariant when an edge is added. The results extend those of Fan [Linear and Multilinear Algebra 50 (2002) 133] for general graphs, and So [Linear and Multilinear Algebra 46 (1999) 193] for simple graphs.

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1. Introduction

Let $G = (V, E)$ be a *mixed graph* of order n with vertex set $V = V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = E(G) = \{e_1, \ldots, e_m\}$, which is obtained from a undirected graph by orienting some of its edges. G is allowed here to have multi-edges and multi-loops (that is, a vertex may be incident with more than one loops). Then some edges of G have a special head and tail, while others do not. The notion of mixed graph generalize both the classical approach of orienting all edges [3] and the unoriented approach [10]. It is important to stress, however, that the mixed graphs in this paper are considered the underlying (undirected) graphs in terms of defining degrees, cycles, connectedness, etc. Since the oriented loops play no useful role in our discussion, we will assume that all loops of G are unoriented.

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To avoid confusions, we call the elements $\{u, w\} \in E(G)$ $(u \neq w)$ the edges of G and $\{u, u\} \in E(G)$ the loops of G. The *incidence matrix* of G is the $n \times m$ matrix $M = M(G) = [m_{ij}]$ whose entries are given by $m_{ij} = 1$ if e_j is an unoriented edge incident with v_i or e_j is a loop incident with v_i or e_j is an oriented edge with head v_i, $m_{ij} = -1$ if e_j is an oriented edge with tail v_i, and $m_{ij} = 0$ otherwise. The *Laplacian matrix* of G is defined to be the matrix $L = L(G) = MM^{\mathrm{t}}$, where M^{t} denotes the transpose of M. (The definition here is a little different from that of Bapat et al. [1] as they set $m_{ij} = 2$ when e_j is a loop incident with v_i.) The sign of $e \in E(G)$ is denoted by $\mathrm{sgn}(e)$ and defined to $\mathrm{sgn}(e) = 1$ if e is unoriented and $\mathrm{sgn}(e) = -1$ otherwise. Set $a_{ij} = \sum_{\{v_i, v_j\} \in E(G)} \mathrm{sgn}(\{v_i, v_j\})$ $(i \neq j)$ if v_i and v_j are joined by some edges and $a_{ij} = 0$ else. Then $A(G) = [a_{ij}]$ is called the *adjacency matrix* of G. The degree matrix $D(G)$ of G is the diagonal matrix $\mathrm{diag}\{d(v_1), \ldots, d(v_n)\}$, where $d(v)$ denotes the degree of the vertex v. Note that a loop contributes 1 to the degree of the incident vertex. From the incidence matrix of G, it is easy to see $L(G) = D(G) + A(G)$, and $L(G)$ is symmetric and positive semidefinite so that and its eigenvalues can be arranged as follows: $0 \leqslant \lambda_1(G) \leqslant \lambda_2(G) \leqslant \cdots \leqslant \lambda_n(G)$. We simply call the eigenvalues and eigenvectors of $L(G)$ as those of G respectively. The *Laplacian spectrum* of G is defined by the multi-set $S(G) = \{\lambda_1(G), \lambda_2(G), \ldots, \lambda_n(G)\}$.

Clearly, if G is *all-oriented* (i.e., except loops, all edges of G are oriented), then $L(G)$ is a Laplacian matrix which is consistent with that of a general graph [4,7] (or of a simple graph if G has no loops and multi-edges [16]); and there are a lot of results involved with the relations between its spectrum and numerous graph invariants, such as connectivity, diameter, matching number, isoperimetric number, expanding properties of a graph (see, for example, [8,16,18,19]). For many properties of mixed graphs, one can refer to [1,2,15,21].

Harary and Schwenk [13] studied those simple graphs G such that $A(G)$ has integral spectrum. The analogous problem for $L(G)$ is also interesting [12]. A simple graph G is said to be *Laplacian integral* if $S(G)$ consists entirely of integers. In [11,12], some properties of trees with integral eigenvalues and Laplacian integral graphs were discussed respectively. Merris [17] has shown that the degree maximal graphs are Laplacian integral. In [20], So considered the problem of preserving Laplacian integrality by adding an edge, and gave an equivalent condition for a simple graph with exactly one eigenvalue moving up by an integer and others remaining invariant. Fan [5] introduced the notion of *spectral integral variations* to study the general graphs (i.e. the all-oriented mixed graphs) with all changed eigenvalues moving up by integers by adding an (oriented) edge, and provided a method to construct a new Laplacian integral graph from a known one. If the spectral integral variation of a general graph occurs by adding an (oriented) edge, then it must occur either in one place or in two places [5,20]; and the problem for the former case was solved. Recently, Fan [6] characterizes the degree maximal graphs with the spectral integral variation occurring in two places by adding an (oriented) edge.

In this paper, we discuss the spectral integral variations of mixed graphs by adding an unoriented or oriented edge, and obtain some equivalent conditions for a mixed graph with the spectral integral variation occurring in one place (or equivalently, with exactly one eigenvalue moving up by an integer and others remaining invariant) by adding an edge. The results we obtained extend those of Fan [5] for general graphs and So [20] for simple graphs.

2. Results

A multi-set is one in which the elements are allowed to be the same. For example, the edge set, the neighbourhood of a vertex and the Laplacian spectrum of a mixed graph are all multi-sets. Let S be a multi-set. Denote by $m_S(e)$ the multiplicity of $e \in S$. If there is no confusions somewhere, we will directly write $m(e)$ instead of $m_S(e)$. If $e \in S$ with multiplicity m, denote $e^{(m)} \in S$. If $e \notin S$, then we write $m_S(e) = 0$ or $e^{(0)} \in S$. Define $S_1 \cup S_2 = \{e^{(m)} : m = m_{S_1}(e) + m_{S_2}(e) > 0\}$, $S_1 \cap S_2 = \{e^{(m)} : m = \min(m_{S_1}(e), m_{S_2}(e)) > 0\}$, and $S \backslash S_1 = \{e^{(m)} : m = m_S(e) - m_{S_1}(e) > 0\}$.

Let G be a mixed graph and let e be an (oriented or unoriented) edge or a loop with the incident vertices (vertex) both in G. Note that e may or may not belong to G. Denote by $G + e$ the mixed graph obtained from G by adding a new edge e. Let $T = S(G) \cap S(G + e)$. Arranging the elements of $S(G + e)\backslash T$ and $S(G)\backslash T$ in nondecreasing order, we say that the spectral variation of G is *integral* (or the *spectral integral variation* of G occurs) by adding e if the differences between the elements of $S(G + e)\backslash T$ and $S(G)\backslash T$ in the same places are integral. If the spectral integral variation of G occurs and the cardinality of $S(G)\backslash T$ is k, then we say that the spectral integral variation of G occurs in k places. With the following result, the possible real values of k must be 1 or 2.

Lemma 1. *Let G be a mixed graph of order n and let e be an edge or a loop. Then*

$$\lambda_1(G) \leqslant \lambda_1(G + e) \leqslant \lambda_2(G) \leqslant \cdots \leqslant \lambda_n(G) \leqslant \lambda_n(G + e).$$

Proof. Denote by $K(G) = M(G)^{\mathrm{t}} M(G)$. Then $L(G)$ and $K(G)$ have the same nonzero eigenvalues. Noting that $K(G)$ is the principal submatrix of $K(G + e)$, the result follows from Courant–Fischer theorem [14, Theorem 4.2.11, p. 179]. □

The following lemma is for general graphs [5, Lemma 1], but the proof carries over verbatim for the mixed graphs.

Lemma 2 [5]. *Let G be a mixed graph of order n, and $S(G) = \{\lambda_1(G), \lambda_2(G), \ldots, \lambda_n(G)\}$. Then*

(1) *The spectral integral variation of G by adding an edge e occurs only in the following two cases*:

(1.1) *The spectral integral variation occurs in one place*, *hence*

$$S(G+e) = \big(S(G)\backslash\{\lambda_k(G)\}\big) \cup \big\{\lambda_k(G)+2\big\} \quad \textit{for some } k.$$

(1.2) *The spectral integral variation occurs in two places*, *hence*

$$S(G+e) = \big(S(G)\backslash\{\lambda_k(G), \lambda_l(G)\}\big) \cup \big\{\lambda_k(G)+1, \lambda_l(G)+1\big\}$$
$$\textit{for some } k \textit{ and } l.$$

(2) *The spectral integral variation of G by adding a loop e occurs only in one place*, *therefore*

$$S(G+e) = (S(G)\backslash\{\lambda_k\}) \cup \{\lambda_k+1\} \quad \textit{for some } k.$$

In the following, we adopt the terminology from [9]: for a graph G of order n, and an eigenvector x of G or generally any vector x of $\mathbb{R}^n$, we say x gives a *valuation* of the vertices of G, and for each vertex v_i of G, we associate the number x_i (the ith entry of x), which is the valuation of vertex v_i. Formally, the (vertex) valuation afforded by x is the function $x : V \to \mathbb{R}$ defined by $x(v_i) = x_i$, $1 \leqslant i \leqslant n$. Then for any vector x,

$$x^{\mathrm{t}}L(G)x = \sum_{\substack{u \neq w,\\ \{u,w\}\in E(G)}} (x(u) + \mathrm{sgn}(\{u,w\})x(w))^2 + \sum_{\{u,u\}\in E(G)} x(u)^2, \tag{1}$$

and x is an eigenvector of G corresponding to λ if and only if $x \neq 0$ and for each vertex $u \in V(G)$,

$$(\lambda - d(u))x(u) = \sum_{\substack{u \neq w,\\ \{u,w\}\in E(G)}} \mathrm{sgn}(\{u,w\})x(w). \tag{2}$$

Let $S(G) = \{\lambda_1(G), \lambda_2(G), \ldots, \lambda_n(G)\}$. If the spectral integral variation of G occurs in one place by adding an edge e, then by Lemma 2, $S(G+e) = (S(G)\backslash \{\lambda_k(G)\}) \cup \{\lambda_k(G)+2\}$ for some k. If $\lambda_k(G)+2 \leqslant \lambda_{k+1}(G)$, then the nondecreasing arrangement of the elements of $S(G+e)$ is easily obtained from $S(G)$ only by replacing $\lambda_k(G)$ with $\lambda_k(G)+2$. Otherwise, the nondecreasing arrangement of the elements of $S(G+e)$ is

$$\lambda_1(G) \leqslant \cdots \leqslant \lambda_{k-1}(G) \leqslant \lambda_{k+1}(G)$$
$$\leqslant \cdots \leqslant \lambda_l(G) < \lambda_k(G)+2 \leqslant \lambda_{l+1}(G) \leqslant \cdots \leqslant \lambda_n(G).$$

So after a nondecreasing arrangement, G and $G+e$ have some common eigenvalues in the same places.

Lemma 3. *Let G be a mixed graph of order n and let $e = \{u, w\}$ be an edge. If $\lambda_r(G) = \lambda_r(G+e) = \lambda_r$, $r = 1, 2, \ldots, p$, and (or) $\lambda_{n-s}(G) = \lambda_{n-s}(G+e) = \lambda_{n-s}$, $s = 0, \ldots, q$, then for each r and (or) s ($r = 1, 2, \ldots, p$, $s = 0, \ldots, q$), G and $G+e$ have the same orthonormal eigenvectors v_r corresponding to λ_r and* *(or)* ν_{n-s} *corresponding to* λ_{n-s}, *respectively*, $\nu_r(u) + \operatorname{sgn}(e)\nu_r(w) = 0$ *and (or)* $\nu_{n-s}(u) + \operatorname{sgn}(e)\nu_{n-s}(w) = 0$.

Proof. Let $S_1 = \{x \in \mathbb{R}^n : x^{\mathrm{t}}x = 1\}$. There exists $\nu_1 \in S_1$ such that $L(G+e)\nu_1 = \lambda_1(G+e)\nu_1$. By the equality (1) and Rayleigh–Ritz theorem [14, Theorem 4.2.2],

$$\begin{aligned}\lambda_1(G+e) &= \nu_1^{\mathrm{t}} L(G+e)\nu_1 \\ &= \nu_1^{\mathrm{t}} L(G)\nu_1 + (\nu_1(u) + \operatorname{sgn}(e)\nu_1(w))^2 \geqslant \lambda_1(G),\end{aligned}$$

If $\lambda_1(G) = \lambda_1(G+e)$, then $\nu_1(u) + \operatorname{sgn}(e)\nu_1(w) = 0$, $\nu_1^{\mathrm{t}} L(G)\nu_1 = \lambda_1(G)$. So ν_1 is a unit eigenvector of G corresponding to $\lambda_1(G)$.

Let $S_2 = \{x \in \mathbb{R}^n : x \perp \nu_1, x^{\mathrm{t}}x = 1\}$. There exists $\nu_2 \in S_2$ such that $L(G+e)\nu_2 = \lambda_2(G+e)\nu_2$. Then

$$\begin{aligned}\lambda_2(G+e) &= \nu_2^{\mathrm{t}} L(G+e)\nu_2 \\ &= \nu_2^{\mathrm{t}} L(G)\nu_2 + (\nu_2(u) + \operatorname{sgn}(e)\nu_2(w))^2 \geqslant \lambda_2(G).\end{aligned}$$

If $\lambda_2(G) = \lambda_2(G+e)$, then $\nu_2(u) + \operatorname{sgn}(e)\nu_2(w) = 0$, $\nu_2^{\mathrm{t}} L(G)\nu_2 = \lambda_2(G)$. So ν_2 is a unit eigenvector of G corresponding to $\lambda_2(G)$. The discussions on $\lambda_3, \ldots, \lambda_p$ and $\lambda_n, \ldots, \lambda_{n-q}$ are similar, and the result follows. □

Let G be a mixed graph of order n and let $e = \{u, w\}$ $(u, w \in V(G))$ be an (oriented or unoriented) edge or a loop. The *incidence vector* corresponding to e is denoted by I_e and defined to be the real vector with exactly two nonzero entries: $I_e(u) = \operatorname{sgn}(e)I_e(w) = 1$ if e is unoriented or u is the head of the oriented edge e and -1 otherwise. Note that I_e is the column vector of the incidence matrix $M(G)$ or $M(G+e)$ corresponding to the edge e depending on whether or not e is an edge or a loop of G and if e is a loop then I_e is a vector with exactly one nonzero entry $I_e(u) = 1$. The following theorem give a characterization of the spectral integral variation of a mixed graph occurring in one place by means of incidence vector.

Theorem 1. *Let G be a mixed graph of order n and let $e = \{u, w\}$ be an edge. Then the spectral integral variation of G occurs in one place by adding the edge e if and only if the incidence vector I_e is an eigenvector of G.*

Proof. If I_e is an eigenvector of G corresponding to the eigenvalue λ, then by the equality (2), for any vertex v $(v \neq u, v \neq w)$,

$$(\lambda - d(v))0 = \sum_{\{v,u\} \in E(G)} \operatorname{sgn}(\{v,u\})I_e(u) + \sum_{\{v,w\} \in E(G)} \operatorname{sgn}(\{v,w\})I_e(w), \tag{3}$$

and

$$(\lambda - d(u))I_e(u) = \sum_{\{w,u\} \in E(G)} (\operatorname{sgn}(\{u,w\})I_e(w), \tag{4}$$

$$(\lambda - d(w))I_e(w) = \sum_{\{u,w\}\in E(G)} (\mathrm{sgn}(\{w,u\})I_e(u). \tag{5}$$

The above three equalities also hold if we replace G by $G+e$, $d(u)$ by $d(u)+1$, $d(w)$ by $d(w)+1$ and λ with $\lambda+2$ since $I_e(u) = \mathrm{sgn}(e)I_e(w)$. So $\lambda+2$ is an eigenvalue of $G+e$ with the corresponding eigenvector I_e. For any eigenvector ν of G orthogonal to I_e, we have that $\nu(u) + \mathrm{sgn}(e)\nu(w) = 0$ and hence

$$L(G+e)\nu = L(G)\nu + (\nu(u) + \mathrm{sgn}(e)\nu(w))\mu = L(G)\nu,$$

which implies that ν is also an eigenvector of $G+e$, where μ is a vector with only two nonzero entries $\mu(u) = \mathrm{sgn}(e)\mu(w) = 1$ or -1. Therefore, each eigenvalue with corresponding eigenvector orthogonal to I_e is also an eigenvalue of $G+e$ with the same eigenvector. The sufficiency holds.

Assume that the spectral integral variation of G occurs in one place by adding the edge e. By the discussion prior to Lemma 3, the arrangement of the eigenvalues of $G+e$ has two cases. We begin with the first case. By Lemma 3, G, $G+e$ have $n-1$ common orthonormal eigenvectors corresponding the eigenvalues in $S(G)\backslash\{\lambda_k(G)\}$. Hence, there is a same unit eigenvector ν_k corresponding to $\lambda_k(G) = \lambda_k$, $\lambda_k(G+e) = \lambda_k + 2$. And

$$\lambda_k + 2 = \nu_k^{\mathrm{t}} L(G+e)\nu_k = \nu_k^{\mathrm{t}} L(G)\nu_k + (\nu_k(u) + \mathrm{sgn}(e)\nu_k(w))^2 = \lambda_k + 2.$$

So $(\nu_k(u) + \mathrm{sgn}(e)\nu_k(w))^2 = 2$, which implies $\nu_k(v) = 0$ for any $v \notin \{u, w\}$, and $\nu_k(u) = \mathrm{sgn}(e)\nu_k(w) = 1/\sqrt{2}$ (or $-1/\sqrt{2}$). The necessity follows.

For the second case, $\lambda_r(G) = \lambda_r(G+e)$, $r = 1, 2, \ldots, k-1, l+1, l+2, \ldots, n$. By Lemma 3, G, $G+e$ have same orthonormal eigenvectors ν corresponding to the above eigenvalues, which satisfies $\nu(u) + \mathrm{sgn}(e)\nu(w) = 0$. Let S_* be the complement of the subspace spanned by the eigenvectors above. In the space S_*, there exist eigenvectors of G corresponding to eigenvalues $\lambda_k = \lambda_k(G) \leqslant \lambda_{k+1}(G) \leqslant \cdots \leqslant \lambda_l(G)$, and eigenvectors of $G+e$ corresponding to $\lambda_{k+1}(G) \leqslant \cdots \leqslant \lambda_l(G) < \lambda_k(G) + 2$. Let $y = \frac{1}{\sqrt{2}} I_e$. It is easily seen that $y \in S_*$. We have

$$\begin{aligned}
\lambda_k + 2 &= \max_{x\in S_*} x^{\mathrm{t}} L(G+e)x \\
&\geqslant y^{\mathrm{t}} L(G+e)y = y^{\mathrm{t}} L(G)y + (y(u) + \mathrm{sgn}(e)y(w))^2 = y^{\mathrm{t}} L(G)y + 2 \\
&\geqslant \min_{x\in S_*} x^{\mathrm{t}} L(G)x + 2 = \lambda_k + 2.
\end{aligned}$$

Therefore, $y^{\mathrm{t}} L(G)y = \lambda_k$, and y is a unit eigenvector of G corresponding to $\lambda_k(G)$. The result follows. □

Let G be a mixed graph and let e be an edge. Denote by e^c the edge obtained from e by orienting it if e is unoriented or unorienting it if e is oriented. It is easily seen that $\mathrm{sgn}(e)\mathrm{sgn}(e^c) = -1$. We use $e_1^{e_2}$ to denote the edge e_1 whose sign is the same as that of e_2.

Lemma 4. *Let G be a mixed graph and let $e = \{u, w\}$ be an edge. Then I_e is an eigenvector of G if and only if $d(u) = d(w)$ and for any vertex $v \notin \{u, w\}$,* $\mathrm{sgn}(e)[m(\{v, u\}^e) - m(\{v, u\}^{e^c})] + [m(\{v, w\}^e) - m(\{v, w\}^{e^c})] = 0.$

Proof. Suppose I_e is an eigenvector of G. Then we have equalities (3)–(5) in the proof of Theorem 1. Without loss of generality, let $I_e(u) = 1$, $I_e(w) = \mathrm{sgn}(e)$. Then from (3), for any vertex $v \notin \{u, w\}$,

$$\mathrm{sgn}(e)[m(\{v, u\}^e) - m(\{v, u\}^{e^c})] + [m(\{v, w\}^e) - m(\{v, w\}^{e^c})] = 0.$$

Similarly from (4) and (5),

$$\begin{aligned}\lambda &= d(u) + [m(\{u, w\}^e) - m(\{u, w\}^{e^c})]\\ &= d(w) + [m(\{u, w\}^e) - m(\{u, w\}^{e^c})].\end{aligned}$$

So $d(u) = d(w)$. The necessity holds. Retracing our proof, the equalities (1)–(3) hold, and hence I_e is one of eigenvectors of G. □

By the proof of Theorem 1 and Lemma 4, we have the following results.

Corollary 1. *Let $G = (V, E)$ be a mixed graph of order n and let $e = \{u, w\}$ be an edge. If the spectral integral variation of G occurs in one place by adding the edge e, then the changed eigenvalue of G is the sum of $d(u)$ (or $d(w)$) and $m(\{u, w\}^e) - m(\{u, w\}^{e^c})$ with the incidence vector I_e as a corresponding eigenvector.*

Corollary 2. *Let $G = (V, E)$ be a mixed graph of order n and let $e = \{u, w\}$ be an edge. Then the following conditions are equivalent*:

(1) *The spectral integral variation of G occurs in one place by adding e.*
(2) *The incidence vector I_e is an eigenvector of G.*
(3) *$d(u) = d(w)$ and for any vertex $v \notin \{u, w\}$,* $\mathrm{sgn}(e)[m(\{v, u\}^e) - m(\{v, u\}^{e^c})] + [m(\{v, w\}^e) - m(\{v, w\}^{e^c})] = 0.$

By (3) of Corollary 2, we also obtain the following result.

Corollary 3. *Let G be a mixed graph with all edges unoriented or all edge (except loops) oriented. Then the spectral integral variation of G occurs in one place by adding an edge $e = \{u, w\}$ if and only if*

(1) *u and w have the same multiplicities of loops and each vertex except u or w has the same multiplicities of edges joining u and joining w if e is oriented.*
(2) *u and w have the same multiplicities of loops and each vertex except u or w has no edges joining u or w if e is unoriented.*

Let G be a mixed graph and let C be a cycle of G. C is called *nonsingular* if the submatrix of $M(G)$ with rows indexed by the vertices of C and column indexed by the edges of C is nonsingular. By the proof of [1, Lemma 1], C is nonsingular if and only if C contains an odd number of unoriented edges. A mixed graph is called *quasi-bipartite* if it does not contain a nonsingular cycle. Obviously, a quasi-bipartite mixed graph G can not have loops, and hence the incidence matrix $M(G)$ and the Laplacian matrix $L(G)$ are respectively the same as those defined in [1]. So, for the incidence matrices and Laplacian matrices defined in this paper, [1, Theorem 4] still holds by observing its proof. We will directly use this theorem in the following discussion.

Note that a *signature matrix* is a diagonal matrix with ± 1 along the diagonal. Denote by $\overrightarrow{G}$ the all-oriented graph obtained from the mixed graph G by orienting all of its edges except loops.

Lemma 5. *Let G be a connected mixed graph of order n. Then $L(G)$ is singular if and only if G is quasi-bipartite.*

Proof. Suppose that G is quasi-bipartite. Then G has no loops, and by [1, Theorem 4] there exists a signature matrix D such that $DL(G)D^{\mathrm{t}} = L(\overrightarrow{G})$. Clearly, $L(\overrightarrow{G})$ is singular, so is $L(G)$. Conversely, let $x = (x_1, \ldots, x_n)^{\mathrm{t}} \neq 0$ such that $L(G)x = 0$. We assert that G has no loops and each entry of x is nonzero; otherwise by the equality (1), $x = 0$ since $x^{\mathrm{t}}L(G)x = 0$ and G is connected. Then

$$0 = x^{\mathrm{t}}L(G)x = \sum_{\substack{u \neq w,\\ \{u,w\} \in E(G)}} (x(u) + \mathrm{sgn}(\{u, w\})x(w))^2,$$

and hence each term of the sum is zero. So there exists a real number $k > 0$ such that $|x(u)| = |x(v)| = k$ for any vertices $u, v \in V(G)$. Then we have a partition $V(G) = V_1 \cup V_2$, where $V_1 = \{u : x(u) = k\}$ and $V_2 = \{u : x(u) = -k\}$. Also by the above equality, one can find that every edge between V_1 and V_2 is unoriented and every edge within V_1 or V_2 is oriented. By [1, Theorem 4] again, G is quasi-bipartite. □

Theorem 2. *Let G be a mixed graph obtained from a quasi-bipartite graph H by adding some or no loops. Then the spectral integral variation of G occurs in one place by adding an edge $e = \{u, w\}$ if and only if*

(1) *G has same property as those in* (1) *of Corollary* 3 *if $H + e$ is quasi-bipartite.*
(2) *G has same property as those in* (2) *of Corollary* 3 *if $H + e$ is not quasi-bipartite.*

Proof. If $H + e$ is quasi-bipartite, by [1, Theorem 4], there exists a signature matrix D such that $L(H + e) = DL(\overrightarrow{H + e})D^{\mathrm{t}}$, and hence $L(G + e) = DL(\overrightarrow{G + e})D^{\mathrm{t}}$, $L(G) = DL(\overrightarrow{G})D^{\mathrm{t}}$. So the spectral integral variation of G occurring in one place by adding e if and only if that of $\overrightarrow{G}$ occurring in one place by adding the oriented edge e. Hence (1) follows from Corollary 3.

If $H + e$ is not quasi-bipartite, similarly, we have $L(G) = DL(\overrightarrow{G})D^{\mathrm{t}}$, where D is a signature matrix. Let $(G + e)'$ be the graph satisfying $L(G + e) = DL((G + e)')D^{\mathrm{t}}$. Except loops and the adding edge e, all edges of $(G + e)'$ are oriented. Since $H + e$ is not quasi-bipartite, $(G + e)'$ cannot be $\overrightarrow{G + e}$, and hence e is unoriented in $(G + e)'$. Then the spectral integral variation of G occurring in one place by adding e if and only if that of $\overrightarrow{G}$ occurring in one place by adding the unoriented edge e. The result follows. □

Remark. In [5], the author uses *Faria vector* (the vector with exactly two nonzero entries: 1 and -1) to characterize the spectral integral variation of a general graph by adding an (oriented) edge. The result [5, Theorem 2] generalizes that of So [20, Theorem 2.2] for simple graphs and is listed as follows:

> *The spectral integral variation of a general graph G occurs in one place by adding an (oriented) edge $e = \{u, w\}$ if and only if G has a Faria vector x with $x(u) = -x(w) = 1$ as an eigenvector.*

So Theorem 1 is an extension of above, and the consequent results generalize the corresponding parts of those in [5]. In addition, the equivalent condition for a general graph G with the spectral integral variation occurring by adding a loop e [5, Theorem 3] can also be interpreted as follows: the incidence vector I_e is an eigenvector of G. The question quite naturally arises on how to characterize the mixed graph with the spectral integral variation occurring (necessarily at one place) by adding a loop; this problem will be addressed in a subsequent paper.

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